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LETTER TO THE EDITOR

Ultradiffusion: the relaxation of hierarchical systems

B A Huberman and Michel Kerszberg†
Xerox Palo Alto Research Center, Palo Alto, CA 94304, USA

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Abstract. We show that the dynamics of relaxation in hierarchical structures leads to an anomalous decay process which we term ultradiffusion. Using renormalisation group techniques, we explicitly calculate the behaviour of the autocorrelation function in a one-dimensional hierarchical system and show its universal power law decay as a function of temperature. We also demonstrate analytically the emergence of a hierarchy of time scales. This phenomenon is relevant to systems ranging from macromolecules to computing structures.

Complexity in natural and artificial systems often manifests itself in hierarchical fashion: at any given level of the system the effect of the lower echelons can, for all practical purposes, be integrated over while the larger scale structures are essentially frozen and act as static constraints. The following example of this behaviour was introduced by Simon and Ando (1961) while studying the long-time behaviour of economic systems. Consider the thermalisation of a building partitioned by thick walls, the large rooms being in turn divided by thinner screens which in turn contain rooms partitioned by thinner ones. If an initial temperature gradient is established in the building, the approach to equilibrium through partitions with different thermal conductivities will lead to diffusion coefficients whose actual values depend on time. This process is reminiscent of the non-ergodic behaviour exhibited by systems with a hierarchy of energy barriers, as reported in molecular diffusion on complex macromolecules by Austin *et al* (1975), in macromolecule motions by Levitt (1982), some one-dimensional superionic conductors studied by Boyce and Huberman (1979), and for the spin glass systems investigated by Sompolinsky (1981) and Palmer *et al* (1984). Other systems with a large multiplicity of time scales might range from computing structures, as examined by Huberman and Hogg (1984), to evolutionary processes which were discussed by Simon (1962) many years ago.

Mathematically, Markovian hierarchical systems are described by stochastic transition matrices which are near-decomposable, and solving for their dynamics amounts to an efficient evaluation of their eigenvalues. The spectrum of such matrices always contains the value one, corresponding to the state S of complete equilibrium; i.e.; $MS(t) = S(t+1) = S(t)$, with M the transition matrix. Moreover, the smaller eigenvalues cluster densely around one, leading to abnormal transient phenomena.

A common feature of hierarchical systems is that they can be characterised by an ultrametric topology, as described by Bourbaki (1966), and postulated for the spin glass problem by Mézard *et al* (1984); i.e. a distance d can be defined so that any

† Present address: KFA-Institut fur Festkorperforschung, D-5170 Julich, West Germany.

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triplet of states can be labelled in such a way that their respective distances obey,

$$d(a, b) = d(b, d) \geqslant d(c, a). \tag{1}$$

In what follows, we will construct a simple one-dimensional model which possesses such a topology and study its dynamical properties by explicit renormalisation. We also show that for thermally activated processes, the relaxation of the autocorrelation function obeys a universal algebraic law which we term ultradiffusion. In particular, its long-time behaviour is characterised by an effective dimensionality which is temperature dependent, leading to an anomalous low-frequency spectrum reminiscent of the $1/f$ noise type of phenomena observed in a variety of systems reviewed by Hooge *et al* (1981). Finally, we discuss the relevance of ultradiffusion to the self-repairing computing arrays recently introduced by Huberman and Hogg.

Consider a particle hopping from cell to cell on a line with energy barriers distributed in a hierarchical way as shown on figure 1(*a*). The barriers are labelled by ε_i, the probability that they will be crossed in unit time. Hence, the taller the barrier the smaller is ε. It is easy to see that such a system has an ultrametric topology, i.e. it obeys equation (1) if we define $d(a, b)$ to be the largest barrier between cells a and b. As illustrated in figure 1(*b*) to travel between two points in the top branches of the tree without leaving it, one must go down by a number of levels equal to the ultrametric distance separating the points. Furthermore, we take the ratio of two successive barrier heights, $R = \varepsilon_{i+1}/\varepsilon_i$, to be much smaller than one. With this assumption the system displays a well defined hierarchy of relaxation times, which we now evaluate.

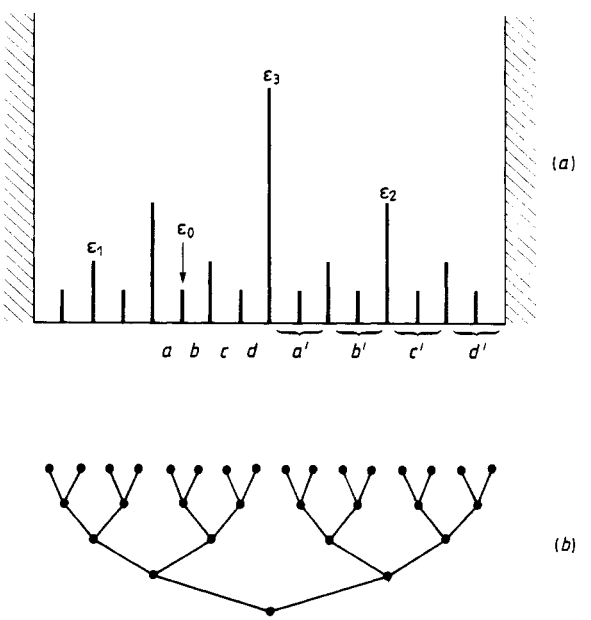

Figure 1. (*a*) Hierarchical array of barriers over which a particle diffuses. The barriers are labelled by ε_i, the probability per unit time that they will be crossed; ε_i is small for a tall barrier. The hierarchy may or may not extend down to infinity. (*b*) Ultrametric structure: to travel between two points in the top branches of the tree without leaving the tree, one must go down by a number of levels equal to the ultrametric distance separating the points.

The basic idea of our renormalisation scheme is to observe that after a length of time $\tau_0 \simeq 1/\varepsilon_0$ the statistical distribution on both sides of the lowest barriers is roughly equalised. Once this happens these barriers can be safely ignored as there is no net flux across them. However, the net flux through the next higher barrier, ε_1, is renormalised in this process. We must therefore write an approximate master equation for the populations of four cells, a, b, c, d in figure 1(a):

$$\begin{aligned} dP_a/dt &= -\varepsilon_0 P_a + \varepsilon_0 P_b + \delta_a \\ dP_b/dt &= \varepsilon_0 P_a - (\varepsilon_0 + \varepsilon_1) P_b + \varepsilon_1 P_c \\ dP_c/dt &= \varepsilon_1 P_b - (\varepsilon_0 + \varepsilon_1) P_c + \varepsilon_0 P_d \\ dP_d/dt &= \varepsilon_0 P_c - \varepsilon_0 P_d + \delta_d \end{aligned} \tag{2}$$

where δ_a and δ_d are correction terms which are at least of order R^2 (see below). From these equations, we can compute the probability that either cell c or d be occupied after a time $A\tau_0$, with $A = O(1)$, assuming that the particle is initially in a or b. We obtain

$$\begin{aligned} P_c + P_d = \tfrac{1}{2} + [\![&-[1+(1+R^2)]^{1/2} \exp\{[-\varepsilon_1 - \varepsilon_0 + (\varepsilon_1^2 + \varepsilon_0^2)^{1/2}] A\tau_0\} \\ &+ [1-(1+R^2)]^{1/2} \exp\{[-\varepsilon_1 - \varepsilon_0 - (\varepsilon_1^2 - \varepsilon_0^2)^{1/2}] A\tau_0\}]\!]\tfrac{1}{4}(1+R^2)^{-1/2} \end{aligned} \tag{3}$$

which in the limit of small R becomes

$$P_c + P_d \simeq A\tau_0 \frac{R[1-R+(1+R^2)^{1/2}]}{4(1+R^2)^{1/2}} \varepsilon_0 \tag{4}$$

with $\tau_0 = \varepsilon_0^{-1}$. Hence, the approximate effective value of ε_1 (the transition rate) is given by

$$\varepsilon_1' = \frac{1-R+(1+R^2)^{1/2}}{4(1+R^2)^{1/2}} \varepsilon_1. \tag{5}$$

In order to implement the renormalisation procedure for the higher barriers, we first consider ε_2 and the eight cells adjacent to the barrier. Defining block cells a', b', c', d' (see figure 1(a)) and replacing in equation (2) ε_1 by ε_2 and ε_0 by the renormalised ε_1 as given by equation (5), we obtain

$$P_c + P_d \simeq \tfrac{1}{2} A\tau_0 \varepsilon_2 \tag{6}$$

which defines a renormalised transition rate ε_2' as

$$\varepsilon_2' = \tfrac{1}{2}\varepsilon_2. \tag{7}$$

Equation (7) describes a simple geometrical effect: while on each side of the barrier there is only one block cell from which the particle can actually jump over ε_2, it may be found in any of two blocks with equal probability. Similarly, by considering blocks containing 2^i cells clustered in groups of four around a barrier ε_i, and with, at the level $i-1$, two barriers ε_{i-1}', we can derive for ε_i' an expression analogous to equation (7). Denoting the iteration number by superscripts, and relabelling $\varepsilon_i \to \varepsilon_i^{(0)}$, $\varepsilon_i' \to \varepsilon_{i-1}^{(1)}$, leads to the final recursion relations, which to order R^2 are given by

$$\begin{aligned} \varepsilon_0^{(i+1)} &= \frac{\varepsilon_1^{(i)}\{1-\varepsilon_1^{(i)}/\varepsilon_0^{(i)} + [1+(\varepsilon_1^{(i)}/\varepsilon_0^{(i)})^2]^{1/2}\}}{4[1+(\varepsilon_1^{(i)}/\varepsilon_0^{(i)})^2]^{1/2}} \\ \varepsilon_n^{(i+1)} &= \tfrac{1}{2}\varepsilon_{n+1}^{(i)} \qquad n \geqslant 1. \end{aligned} \tag{8}$$

These equations describe the renormalisation of the relaxation time at each step i of the iteration†. A single quantity, ε_0, replaces 2^{N-i} nearly degenerate eigenvalues of the original transition matrix, where N is the total number of levels in the hierarchy. We should also point out that although, in principle, one might want to write more accurate recursion relations by solving at each level a higher order differential system, it is unnecessary to do so since the lowest order transformation is already quite accurate. For example, for $R = 10^{-1}$, $\varepsilon_0 = 10^{-6}$, and with five levels in the hierarchy, the closest eigenvalue to 1 is $1 - 1.18 \times 10^{-7}$ (evaluated by direct numerical diagonalisation) while the recursion relations yield $1 - 1.17 \times 10^{-7}$.

We now proceed to compute the autocorrelation function, which for a large system is approximately position independent. Since at each time scale the probability of finding the particle effectively spreads over a region twice the original, we may write for the autocorrelation function P_{aa}

$$P_{aa} = \sum_{i=0}^{N} \exp(-\nu\tau_i)/2^{i+1} \tag{9}$$

where $\tau_i = 1/\varepsilon_0^{(i)}$. Assuming that at a given time t the exponentials corresponding to $\tau_i > t$ are still of order one, while those with $\tau_i < t$ have decayed to zero, one obtains

$$P_{aa} = (\tfrac{1}{2})^{i+1} - (\tfrac{1}{2})^{N}. \tag{10}$$

In this equation, i is related to t by

$$t \simeq A(R/2)^{1-i}4R^{-1}/(2-R) \simeq A(R/2)^{1-i}(2R^{-1}+1)$$

with $A = O(1)$. Inverting this relation, and assuming a thermal activation picture (i.e.; $R \propto \exp(\text{constant}/T)$, we obtain an expression applicable to a number of physical systems, i.e.

$$P_{aa} = -(\tfrac{1}{2})^{N} + \left(\frac{A[1+\frac{1}{2}\exp(-\text{constant}/T)]}{2t\exp(\text{constant}/T)}\right)^{T\ln 2/(T\ln 2+\text{constant})} \tag{11}$$

This formula explicitly displays the algebraic nature of the decay and the temperature dependence of both the exponent and prefactor. It also shows how in hierarchical systems the exponent, which plays the role of an effective dimensionality, tends to zero as the temperature is lowered, leading to an anomalous low-frequency noise spectrum.

We have verified our calculations by numerically simulating diffusion in hierarchical structures of the type discussed here. For the cases $R = e^{-2}$ and $R = e^{-2.7}$, we find that the autocorrelation function behaves as $t^{-0.24}$ and $t^{-0.19}$, respectively. This shows that, in spite of the fact that R is not very small, the exponents agree with those given by equation (11) to within 6%. It should also be mentioned that recent computer simulations of spin glasses have yielded a power law decay of the remanent magnetisation, with a temperature-dependent exponent (Kinzel 1985).

The methods we have developed can also be used to compute other quantities such as the moments of the Green function, e.g. the average distance travelled by the particle. The latter is given by

$$d = \sum_{i=0}^{N} \operatorname{sgn}(i)[1-\exp(-t/\tau_i)]/2^{N-i+1} \tag{12}$$

† It can be shown (to be published) that equation (8) is accurate to $O(R^2)$ in providing a upper bound for the relaxation time, in spite of the fact that the neglected terms in equation (2), δ_a and δ_d, are themselves of this order.

where sgn(i) denotes the direction (right = +1, left = −1) of the particle net displacement at scale i, and which depends on its starting point. (The quantity sgn(i) can be thought of as equal to $1-2x_2^{(i)}$, where $x_2^{(i)}$ is the ith bit after the 'decimal' point in the binary representation of the starting abscissa x). We thus see that, in general, a given particle has a net average motion, even in the absence of a force acting on it. However, an ensemble of particles will have no net motion since in that case equation (12) has to be averaged over all possible bit sequences. As a final remark we note here that $\langle d^2 \rangle$, with brackets denoting ensemble average grows with a temperature-dependent algebraic exponent which is the negative of the one for the autocorrelation function; this can be seen by squaring equation (12) and performing the averages within the same approximations that led to equation (11).

We now discuss the relevance of our results to layered computing structures of the type studied by Huberman and Hogg (1984). Such architectures, which display remarkable immunity to errors, can be shown to have an ultrametric topology. The ultrametric distance between two inputs is now defined as the number of layers that are affected when the inputs are interchanged.

Because of the existence of such topology, layered computing arrays are good candidates for the observation of ultradiffusion. Consider a possible output state of such a computer. To this one final state there correspond several states in the preceding layer: thus a tree or attractor is attached in state space to every possible output of the array. If the input is allowed to diffuse because of noise, one can monitor the state of the machine whenever it returns to its initial output. This amounts to a random walk at the top of the attractor, with changes of state at each step reaching a depth equal to the ultrametric distance between successive inputs. We have numerically simulated a layered machine where each gate in a given layer receives three input bits, one from the gate immediately above it, and two from the latter's neighbours, and with its output set to 0 or 1 according to a majority rule at the input level. As the top layer inputs are made to diffuse by bit exchange, we let the output wander while simultaneously computing its autocorrelation function. We then observed that it decays algebraically in time, with an effective dimensionality which decreases with the number of layers. Next, we singled out those configurations which produced the same output and computed the autocorrelation function of the nth layer, which once again exhibited algebraic decay, the signature of ultradiffusion. (Notice that increasing the value of n is equivalent to carrying out one more step of the renormalisation transformations given by equation (8)).

In summary, we have introduced and studied a simple model of diffusion in hierarchical structures. The model displays interesting dynamics, such as algebraic decay of autocorrelations and temperature-dependent dimensionality, which should be observable in a number of systems. We believe that the widespread occurrence of hierarchical organisation in nature and artificial systems gives our results a significance which reaches beyond this simple example.

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